INVESTIGATIONS RELATIVE TO THE PROBLEM FOR CLEARING THE APPARENT DISTANCE OF THE MOON FROM THE SUN, OR A STAR, FROM THE EFFECTS OF PARALLAX AND REFRACTION, AND AN EASY AND CONCISE METHOD POINTED OUT.

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THE importance of this problem in the practice of the lunar method of finding the longitude at sea, affords a sufficient excuse for the various solutions that have been offered. Seamen require methods both concise, and not embarrassed by distinction of cases; they are often little able or willing to practise any other. Of the two classes into which the solutions of this problem may be divided, the one in which the correction of the distance is given and not the corrected distance itself, is necessarily embarrassed by a distinction of cases, although otherwise concise, as only requiring tables to a few places of decimals. In the other class are found solutions which do not require any distinction of cases; among such solutions, therefore, it should seem those are to be selected which can be most readily adapted to practice

by precluding the necessity or reducing the labour of computing proportional parts, which is the great inconvenience of methods of this class.

Notwithstanding, however, the inconvenience arising from the distinction of cases, some very eminent mathematicians have exerted themselves in investigating the correction of the distance. The first method by Dr. Maskelyne, the methods by Mr. Lyons, Mr. Witchell, and Mr. Cavendish, are of this class, as are also several investigated with his usual ingenuity by M. Delambre, * one lately given by M. Legendre, † and one by M. Bowditch, of which an account is given by M. Delambre. ‡

The practical conveniences in none of these methods seem to counterbalance the inconveniences, and comparing them with the solutions in which the corrected distance is obtained, several of the latter appear better adapted for general use.

An improvement of Mr. Dunthorne's solution, given in the second edition of the requisite tables, seems to have been the first in which the distinction of cases was avoided, and which was at the same time sufficiently concise to be preferred to the solutions of the other class: yet, in this method, the computation of the parts for seconds was very troublesome. This rule was farther improved by Dr. Maskelyne, in his most useful preface to Taylor's Logarithmic Tables, and those tables, extending to seconds, obviated the inconvenience

* Conn. de Temps, Ann. 12 & 14.

† Mem. Nat. Inst.

‡ Conn. des Temps, 1808.

venience from proportional parts. But Taylor's tables cannot be very generally diffused, and besides they require too particular attention in using them, to be held in that estimation with seamen which is their due.

Mr. Mendoza, to whose exertions nautical astronomy is much indebted, has given, in a valuable paper in the Phil Transactions for 1797, forty different formulas for computing directly the true distance. Of these, that which he selected as best adapted to practice, affords a very concise and plain method of computation, requiring only the addition of five versed sines. It may be doubted, however, whether it is preferable to Mr. Dunthorne's first method, improved by the substitution of versed sines for co-sines, as was done by Dr. Mackay.* Mr. Mendoza's method requires an extensive table for an auxiliary angle (equivalent to the auxiliary table in Mr. Dunthorne's method, and in those derived from it,) and the formation of five different arguments, and also to practise it with convenience, a complete table of versed sines, for at least the semircircle. Mr. Mendoza, anxious to improve still farther the solution of the problem, published his very extensive tables,† by which he reduced the method to be equivalent to taking out and adding together three numbers, and not requiring the formation of arguments. His ingenuity and perseverance in forming, computing, and publishing his tables, are deserving of the greatest praise. But on several accounts those tables will

not

* A Treatise on the Longitude, &c.

† Tab. 11. Collection of tables for Navigation and Nautical Astronomy.

not supersede the use of methods in which shorter tables are employed, although these methods should be somewhat longer in practice, provided they be equally plain. To many persons the necessary expence of the volume will be an objection, many will consider its bulk inconvenient, and many disliking such extensive tables with double arguments, will even prefer the former method which those tables were intended to facilitate.

The method pointed out in this paper is closely connected with Mr. Dunthorne's own method, and still more closely with that method improved by Dr. Mackay. It has no distinction of cases, requires only short tables with single arguments, no proportional parts except what can be taken out by inspection, and these for only one quantity besides the conclusion. If it be thought somewhat longer in practice than that by the extensive tables of Mr. Mendoza, it is not less plain, and, by the assistance of a table particularly arranged, may perhaps be as quickly computed.

As examples are given, it will be easy for those conversant with the practical part of the subject, to form an accurate opinion concerning the method here proposed.

Besides investigations immediately connected with this method, an investigation is also given of a formula for the correction of the distance.

Let H = apparent altitude of the sun or star.
H' = apparent altitude of the moon.
h = true altitude of the sun or star.
h' = true altitude of the moon.

D = apparent distance.
d = true distance.
A = difference of apparent altitudes.
a = difference of true altitudes.
p = horizontal parallax } of sun.
r = refraction of star or }
p' = horizontal parallax } of moon.
r' = refraction }

1. By Spherical Trigonometry

$\frac{\cos A - \cos D}{\cos H \cos H'} = \frac{\cos a - \cos d}{\cos h \cos h'}$ both these quantities being equal to the versed sine of the angle, contained by the two verticals passing through the sun or star, and the moon.

Hence $\cos d = \cos a - \frac{\cos h \cos h'}{\cos H \cos H'}(\cos A - \cos D)$

2. There are two convenient ways of computing the latter term. 1. Forming a table for the whole quantity $\frac{\cos h \cos h'}{\cos H \cos H'}$ 2. Developing this quantity, and then forming tables for the terms of its expansion. The former is adopted in the method of Mr. Dunthorne, and its improvement, and in fact in that of Mr. Mendoza; the latter in the method here proposed.

3. Mr. Dunthorne finds the natural number answering to the Log. $\frac{\cos h \cos h'}{\cos H \cos H'}$ + Log. $(\cos A - \cos D)$* which number subtracted

* Mr. Vince, in his Treatise on practical Astronomy, p. 55, instead of computing Log. $(\cos A - \cos D)$ uses Log. sin $\left(\frac{D+A}{2}\right)$ + Log. sin $\left(\frac{D-A}{2}\right)$ + Log. 2. This is rather more convenient than Mr. Dunthorne's method, as we avoid finding the Logarithm of a number, but this advantage is somewhat lessened by it being necessary to form the arguments $\frac{D+A}{2}$ and $\frac{D-A}{2}$

tracted from the cosine of a, gives the cosine of d. The cosine of an arch greater than 90° being negative, occasions here a distinction of cases. If we use versed instead of co-sines, as Dr. Mackay has done, and put $\frac{\cos h \cos h'}{\cos H \cos H} = N$ we shall have v. sin d = v. sin a + N (v. sin D — v. sin A) where no distinction of cases occur. But there are still inconveniences in this method so improved. An extensive table, with a double argument, is required for the value of log. N (Tab. 9 of the requisite tables) and also logarithmic tables to six or seven places requiring proportional parts. But even with these inconveniences the method, on account of its plainness and conciseness, is valuable, and scarcely yields to any one that has been given.

4. Instead of performing the multiplication by aid of logarithms, it may be done by the assistance of natural sines, and the conclusion reduced to versed sines. Thus, let 2 cos M = N. Then cos d = cos a — 2 cos M (cos A — cos D) = cos a — cos (A + M) — cos (A—M) + cos (D + M) + cos (D—M)

Or reducing this equation to versed sines by substituting for cosine,

v. sin d = v. sin a — v. sin (A + M) — v. sin (A — M) + v. sin (D + M) — v. sin (D — M)

This is one of Mr. Mendoza's formulas; but he prefers a similar one in which the sum of the altitudes occur instead of the difference; the former being somewhat more easily computed than the latter. The inconvenience of this method of multiplying, arises from the formation of the arguments. The

The advantage arises from the simplicity of the conclusion, in reducing the problem to the taking out of five versed sines. As the angle M is always nearly 60°, it is obvious the table of versed sines required by this method must extend to the semicircle.

5. The quantity $\frac{\cos h \cos h'}{\cos H \cos H'}$ $(\cos A - \cos D)$ may be computed by the expansion of $\frac{\cos h \cos h'}{\cos H \cos H'}$. Because $h = H + p \cos H - r$ and $h' = H + h' \cos H' - r'$ we readily obtain by Taylor's theorem.

$$\frac{\cos h}{\cos H} = \frac{\cos(H + p \cos H - r)}{\cos H} = 1 - \sin 1'' (p \sin H - r \tan H) - \tfrac{1}{2} \sin^2 1'' (p \cos H - r)^2 \text{ \&c.}$$

$$\frac{\cos h'}{\cos H'} = 1 - \sin 1'' (p' \sin H' - r' \tan H') - \tfrac{1}{2} \sin^2 1'' (p' \cos H' - r')^2 \text{ \&c.}$$

Hence omitting small terms depending on the sun or star

$$\frac{\cos h \ \cos h'}{\cos H \cos H'} = 1 - \sin 1 \left\{ \begin{matrix} p' \sin H' - r' \tan H' \\ p \sin H - r \tan H \end{matrix} \right\} - \tfrac{1}{2} \sin^2 1'' \left\{ \begin{matrix} (p' \cos H' - r')^2 \\ + r^2 \end{matrix} \right\}$$

Now because the refraction varies nearly as the cotangent of the altitude, and at $45° = 57'', 3$* let

$$r' = 57'', 3 \cot H' - \delta'$$
$$r = 57, 3 \cot H - \delta$$

then

$$\frac{\cos h \cos h'}{\cos H \cos H'} = 1 - \sin 1'' \left\{ \begin{matrix} p' \sin H' + p \sin H - 114'', 6 \\ + \delta' \tan H' + \delta \tan H \end{matrix} \right\} - \tfrac{1}{2} \sin^2 1'' \left\{ \begin{matrix} (p' \cos H' - r')^2 \\ + r^2 \end{matrix} \right\}$$

This

* M. Laplace has proved independently of any hypothesis on the constitution of the atmosphere, that at all heights above 10°, the refraction varies accurately as the cotangent of the altitude (Mecanique Céleste, Tom. iv. p. 269). The tables at the end of this paper are however computed from the tables of refraction, in the requisite tables.

This quantity may with much convenience be put into five small tables.*

The first table may give $\sin 1'' (p' \sin H' - 114'', 6)$ the argument will be $p' \sin H'$ the parallax in altitude answering to the complement of the moon's altitude.

The second table will give $\frac{1}{2} \sin^2 1'' (p' \cos H - r')^2$ and the argument will be the correction of the moon's altitude.

The third table gives $\sin 1'' \delta \tan H + \frac{1}{2} r^2 \sin^2 1''$ and the argument will be the height of the star. This table is only to be used when a star is observed.

The fourth table is to be used when the sun is observed, and gives $\sin 1'' \left\{ \begin{matrix} p \sin H \\ \delta \tan H \end{matrix} \right\} + \frac{1}{2} r^2 \sin^2 1''$ and the argument is the altitude of the sun.

The fifth table gives $\sin 1'' \delta' \tan H'$ and the argument is the altitude of the moon.

Let α, β, γ, ε, represent the quantities given by these tables which are always positive, and then we have

$$\cos d = \cos a - (1 - \alpha - \beta - \gamma - \varepsilon) (\cos A - \cos D)$$

Or lastly,

$$\text{v. sin } d = \text{v. sin } D - \text{v. sin } A - (\alpha + \beta + \gamma + \varepsilon) (\text{v. sin } D - \text{v. sin } A) + \text{v. sin } a.$$

The computation of the above formula will be rendered very plain and short, by a table of versed sines to 120°,† which table, including the addition for seconds, will not contain

* See the tables at the conclusion of this paper, in which all the numbers may be taken out by inspection, and which have all single arguments.

† By using the difference of the altitudes in the above formula, instead of the sum the limit of this table is that of the distance.

contain more than 20 or 25 pages. A table also by which the moon's parallax* may be taken out by inspection, will greatly assist this as well as every other method, and for this method it should be arranged in the manner of tables of sines and cosines, so that the parallax answering to the complement of the moon's altitude, may be found by the side of the parallax itself. But the parallax for any altitude, and for its complement, are so readily found by help of the very valuable table of proportional logarithms in the requisite tables, that the want of the abovementioned table can scarcely be offered as an objection to the practice of this method: particularly when it is considered that the use of proportional logarithms must be familiar to all who use the lunar method, being necessary in a subsequent part of the operation.

The product $(\alpha+\beta+\gamma+\epsilon)$ (v. sin D—v. sin A) will easily be had by a table of logarithms to five places, and indeed to those to whom contracted decimal multiplication is familiar, it will be scarcely worth while to have recourse to a table of logarithms. The odd seconds in D should be reserved, and as H & H′ should be only put down to the nearest minute, seconds will only be used for v. sin *a* and v. sin *d*.

6. Mr. Dunthorne's table for the value of log N (the 9th of the requisite tables) is of considerable extent, and requires, that it may be used with convenience, to be even farther extended

* For low altitudes it seems absolutely necessary to use separate tables for parallax and refraction, instead of a table for the correction of the moon's altitude, on account of the variations of the refraction ascertained by the changes in the barometer and thermometer. Refraction is sometimes changed by $\frac{1}{5}$ of its mean quantity.

extended than it is in the latter editions of the requisite tables. But from the value of N, which has been here given, this table may be reduced to tables of the same conciseness and convenience as those that are above given for N—1.

Thus the common logarithm of the value of N above given

$$= -\frac{\sin 1''}{m}\left\{\begin{array}{l} p'\sin H'+p\sin H \\ -114,''6+\delta'\tan H'+\delta\tan H \end{array}\right\} - \frac{\sin^2 1''}{2\,m}\left\{\begin{array}{l} (p'\cos H'-r')^2 \\ +r^2-p'^2\sin^2 H' \end{array}\right\}$$

where $m = 2{,}03025$

This quantity admits of being put into tables of the same form as those above-mentioned.

Mr. Mendoza's 10th table also may be put into tables of a similar convenient form, supposing his 9th table, instead of containing the correction of the moon's altitude, to contain only the parallax in altitude, and to be arranged in the manner above-mentioned.

For let M be the angle taken out in that table

then $\cos(60°+M) = \frac{1}{2}N$

Whence is deduced

$$M = \frac{1}{\sin 60°}\left\{\begin{array}{l} p'\sin H'+p\sin H \\ -114'',6+\delta'\tan H'+\delta\tan H \end{array}\right\} + \frac{\sin 1''}{\sin 60°}\left\{\begin{array}{l} \frac{1}{2}(p'\cos H'-r)^2 \\ +\frac{1}{2}r^2-\frac{1}{3}(p'^2\sin^2 H') \end{array}\right\}$$

7. The practical rule from the formula given in the last article but one, may be briefly stated as follows:

Practical Rule.

1. Find, by help of a table, the parallax answering to the moon's altitude, and to the complement of the altitude. The latter will be the argument of tab. 1. Or

Compute them by adding the proportional log. of the horizontal parallax to the arithmetical comp. of the log. cos. and log.

log. sin. of alt., the sums will be the prop. logs. of the respective parallaxes.

2. Moon's par.— moon's refrac.=corr. of alt. Take diff. of (corr. of altitude + star's or sun's refraction + moon's alt.) and star's altitude (or sun's alt. + par.) This diff. is the diff. of true altitudes. Find also diff. of app. altitudes.

3. When the sun is observed, add together the numbers in tab. 1, 2, 4, and 5. When a star is observed add the numbers in tab. 1, 2, 3, and 5, log. of this sum (its index being always 3 + number of figures) + log. (v. sin observed dist.—v. sin diff. of observed altitudes), rejecting 10 from the index = log. of a number to be subtracted from the above diff. of versed sines.

4. The remainder + v. sin diff. of true altitudes = v. sin of true distance.

Observations. No distinction of cases occur. No proportional parts but such as are taken out by inspection. The versed sines may considered as whole numbers, the radius being (1,000,000). In taking out the versed sines of the observed distance, the seconds may be reserved and added to the conclusion. Also in finding the log of (v. sin observed dist.—v. sin diff. ob. alt.) the two last figures may be considered as cyphers.

For those conversant in contracted decimal multiplication, the third precept may stand as follows.

3. When the sun is observed, take the sum of the numbers in tab. 1, 2, 4, and 5. When a star, the sum of the numbers in tab. 1, 2, 3, and 5. Find also the excess of the

versed sine of the observed distance, above the versed sine of the difference of observed altitudes. The figures in the above-mentioned sum must be increased to five, if necessary, by prefixing cyphers. Place the first figure of the sum under the third figure of the excess from the right hand, the second figure under the fourth figure of the excess, &c. thus inverting the figures of the sum. The product found according to the method of contracted decimal multiplication, is to be subtracted from the excess.

Example

Example I.

Sun's alt.	19° 4″	observed distance	103° 29′ 27″
Moon's alt.	41 6	horizontal parallax	58 35
Diff. ob. alt.	22 2		

Prop. Log. 58 35	4875		4875	
41° 6′ A C cos	1229	sin	1822	
	6104		6697	38′ 30″ arg. tab. 1.

parallax in alt.		44′	9″		
Moon's ref.		1	5		
Corr. alt.		43	4		145
	41	6		Tab. 1	10497
Sun's refrac.		2	44	Tab. 2	78
	41	51	48	Tab. 4	19
Sun's alt+ par.	19	4	8	Tab. 5	0
diff. true alt.	22	47	40		10739

v. sin 103° 29′	1233162	Log.	10739	8.03100
v. sin 22 2	73034	Log.	1160100	6.06446
	1160128	Log.	12460	4.09546
	12460			
	1147668	Without Logarithms.		
v. sin 22° 47′ ″	78024		11601	
40	74		93701	
	1225766		11601	
			812	
v. sin 103 2 52			35	
27			10	
103 3 19 true dist.			12458	

Example

Example II.

Star's ob. alt.	11° 17′	distance	43° 35′ 42″
Moon's ob. alt.	9 38	horizontal parallax	54 42
diff. ob. alt.	1 39		

Prop. Log.	54 42	5173	5173	
A C cos	9 38	0062	sin 7764	
		5235	1,2937	9′ 9″ arg. tab. 1.

parallax in alt.		53′ 54″			
Moon's refrac.		5 26			
Corr. alt.		48 28			44 }
Star's refrac.		4 40		Tab. 1	2061 }
Moon's alt.	9	38 0		Tab. 2	100
	10	31 8		Tab. 3	11
	11	17		Tab. 5	11
diff. true alt.	0	45 52			2227

v. sin 43° 35′	275628	Log.	2227	7,34772
v. sin 1 39	415	Log.	275200	5,43965
	275213	Log.	613	2,78737
	613			
	274600			

v. sin 0° 45′ ″	86	
52	3	
	274689	
v. sin 43 30 19		
42		
43 31 1 true dist.		

Without Logarithms.

2752
72220
550
55
6
2
613

3. The

8. The above formula will also serve for an easy investigation of an expression, for the correction of the distance.

We have $\cos a - \cos d = (1 - e \sin 1'')(\cos A - \cos D)$

putting $\alpha+\beta+\gamma+\delta = e \sin 1''$

Let $a = A + b$ & $d = D + x$

Then $\cos a = \cos A - b \sin 1'' \sin A - \frac{b^2}{2} \sin^2 1'' \cos A$

$\cos d = \cos D - x \sin 1'' \sin D - \frac{x^2}{2} \sin^2 1'' \cos D$

Hence

$x \sin D - b \sin A + (x^2 \cos D - b^2 \cos A) \frac{1}{2} \sin 1'' = - e (\cos A - \cos D)$

From which equation we obtain

$$x = b\frac{\sin A}{\sin D} - \frac{e(\cos A - \cos D)}{\sin D} - \tfrac{1}{2}\sin 1'' \cot D \left(\frac{b \sin A - e(\cos A - \cos D)}{\sin D}\right)^2 + \tfrac{1}{2} \sin 1'' b^2 \frac{\cos A}{\sin D}$$

and because

$b \sin A + \frac{1}{2} \sin 1'' b^2 \cos A = b \sin (A + \frac{b}{2})$

also $\cos A - \cos D = 2 \sin \left(\frac{A+D}{2}\right) \sin \left(\frac{D-A}{2}\right)$

we have, sufficiently near the truth

$$x = \frac{b \sin \left(A + \frac{b}{2}\right) - 2e \sin \left(\frac{A+D}{2}\right) \sin \left(\frac{D-A}{2}\right)}{\sin D} (c) - \tfrac{1}{2} \sin 1'' c^2 \cot D$$

In this formula c is easily computed by the assistance of proportional logarithms, logarithmic tables, to four or five places of figures, and tables for the value of e similar to the table for N. The term $\frac{1}{2} \sin 1'' c^2 \cot D$ may be had by a table sufficiently convenient. This method of finding the correction of the distance is shorter than those in the requisite tables, and than that of M. Legendre in the Memoirs of the National Institute. It is, notwithstanding the difference in

in the method of investigation, in effect, the same as the second formula of M. Delambre.* And it is not probable that a formula for the correction of the distance more convenient in practice, can be found. But the attention necessary to be given to the signs of the quantities renders it less convenient, than the method in which the corrected distance is obtained.

* Conn. des Temps, pour Ann. 14, p. 31, &c. &c.

Tab.

Tab. I.

" of arg	arg ' & "	' of arg
5	1	
10	2	25
15	3	316
19	4	606
24	5	897
29	6	1188
34	7	1479
39	8	1770
44	9	2061
48	10	2352
53	11	2643
58	12	2934
63	13	3224
68	14	3515
73	15	3806
78	16	4097
83	17	4338
88	18	4679
93	19	4970
97	20	5261
102	21	5552
107	22	5843
112	23	6133
116	24	6424
121	25	6715
126	26	7006
131	27	7297
135	28	7588
140	29	7879
145	30	8170
150	31	8461
155	32	8752
160	33	9043
165	34	9333
169	35	9624
174	36	9915
179	37	10206
184	38	10497
189	39	10788
194	40	11079
199	41	11370
204	42	11661
209	43	11952
213	44	12242
218	45	12533
223	46	12824
228	47	13115
233	48	13046
238	49	13697
242	50	13988
247	51	14279
252	52	14570
257	53	14861
262	54	15151
267	55	15442
272	56	15733
276	57	16024
281	58	16315
286	59	16606
291	60	16896
	61	17187
	62	17478

The argument is the parallax answering to the complement of the apparent altitude of the moon.

Tab. II.

arg. '	
1	0
2	0
3	0
4	1
5	1
6	1
7	2
8	2
9	3
10	4
11	5
12	6
13	7
14	8
15	10
16	11
17	12
18	13
19	15
20	17
21	18
22	19
23	21
24	23
25	26
26	28
27	30
28	32
29	35
30	38
31	40
32	43
33	46
34	49
35	52
36	55
37	58
38	61
39	64
40	68
41	71
42	74
43	78
44	82
45	86
46	90
47	94
48	98
49	102
50	106
51	110
52	114
53	118
54	123
55	128
56	131

The argument is the correction of the moon's apparent altitude.

Tab. III.

arg ° '	
3 0	65
15	57
30	51
3.45	46
4 0	42
30	35
5 0	30
6.0	23
7 0	18
8 0	15
9.0	13
10.0	11
15.0	6
20 0	4
25 0	2
30.0	0
90.0	0

The argument is the star's apparent altitude.

Tab. IV.

arg. ° '	
3 0	67
15	59
30	53
3 45	48
4 0	45
30	38
5 0	33
6 0	27
7.0	22
8 0	20
9.0	19
10 0	18
15.0	17
20 0	19
25 0	20
30.0	21
35 0	24
40 0	27
50.0	33
60.0	37
70 0	41
80 0	42
90.0	43

The argument is the apparent altitude of the sun.

Tab. V.

arg. ° '	
3 0	55
15	48
30	43
3 45	39
4 0	36
30	30
5.0	26
6 0	20
7 0	16
8.0	13
9 0	11
10.0	10
15 0	6
20 0	4
25.0	2
30 0	0
90 0	0

The argument is the apparent altitude of the moon.